Homework 2: Solutions to exercises not appearing in Pressley.

Math 120A

- (1.2.7) Recall that the cycloid is parametrized by $t \mapsto a(t - \sin t, 1 - \cos t)$, and $t = 0$ to $t = 2\pi$ is a complete revolution. The tangent vector $\dot{\gamma}(t)$ is $a(1 - \cos t, -\sin t)$, and has length $||\dot{\gamma}(t)|| = \sqrt{a^2(1-\cos t)^2 + \sin^2 t} = a\sqrt{2 - 2\cos t} = a\sqrt{4\sin^2(\frac{t}{2})} = 2a\sin(\frac{t}{2})$. Ergo the arclength of a single rotation is $s = \int_0^{2\pi} 2a\sin(\frac{t}{2})dt = -4a\cos(\frac{t}{2})|_0^{2\pi} = -4a(-1-1) = 8a$.

- (1.2.9) If $\dddot{\gamma} = 0$, then $\ddot{\gamma}$ is a constant vector $2\mathbf{a}$, implying that $\dot{\gamma} = \mathbf{b} + t2\mathbf{a}$ and $\gamma = \mathbf{c} + \mathbf{b}t + \mathbf{a}t^2$. In particular, every point on γ is the sum of $\mathbf{c}$ and a linear combination of $\mathbf{b}$ and $\mathbf{a}$. We conclude that γ is contained in the plane passing through $\mathbf{c}$ that is parallel to both $\mathbf{a}$ and $\mathbf{b}$ (if one of $\mathbf{a}$ and $\mathbf{b}$ is a multiple of the other, there are infinitely many possible such planes).

- (1.3.6) Since $\dot{\gamma}(t) = (2, \frac{-4t}{(1+t^2)^2})$, γ is certainly regular. Let $\phi(t) = \frac{\cos t}{1+\sin t}$. Then $\phi'(t) = -1(1+\sin t)^2 > 0$, so ϕ is an injection and by the argument with the Inverse Function Theorem mentioned in class, ϕ^{-1} is smooth. Then computation shows that $\gamma \circ \phi$ gives the desired reparametrization.

- (1.5.6) We let $f(t)$ be the function

$$f(t) = \begin{cases} e^{\frac{-1}{t^2}} \quad \&t > 0 \\ 0 \quad \&t \leq 0 \end{cases}$$

This function is smooth (from 131A, say). Now, let $\Theta(t) = \tan(\pi\frac{f(t)}{2})$. This function is smooth and equal to zero on $t \leq 0$. Moreover, $\Theta : (0, \infty) \to (0, \infty)$ is a bijection. Our parametrization of the absolute value curve is

$$\gamma(t) = \begin{cases} (\Theta(t), \Theta(t)) & t \geq 0 \\ (-\Theta(-t), \Theta(t)) & t < 0 \end{cases}$$

All derivatives at zero are zero, and this is a smooth curve. However, $y = |x|$ cannot have a regular parametrization; if it did, it would have a unit speed reparametrization $\tilde{\gamma}(t)$. On $x > 0$, the tangent vector of this curve would necessarily by $\pm\frac{1}{\sqrt{2}}(1, 1)$, so by continuity the tangent vector at 0 would be one of those two vectors. But, on $x < 0$, the tangent vector of this curve would necessarily by $\pm\frac{1}{\sqrt{2}}(-1, 1)$, so by continuity the same holds at zero. These two statements cannot both be true.